THE STABILITY OF EQUILIBRIUM OF FLUID WITHIN A HORIZONTAL CYLINDER HEATED FROM BELOW

(USTOICHIVOST' RAVNOVESIIA ZHIDKOSTI V GORIZONTAL'NOM TSILINDRE, PODOGREVAEVOM SNIZU)

PMM Vol.25, No.6, 1961, pp. 1035-1040

G.Z. GERSHUNI and E.M. ZHUKHOVITSKII
(Perm)

(Received July 24, 1961)

The problem of the stability of equilibrium of fluid in an infinite cylinder heated from below has already been solved by one of the present authors [1]. Only those plane disturbances from equilibrium were considered in which the velocity vector has no component along the axis of the cylinder, all quantities representing the disturbance being independent of the coordinate in the direction of the axis. Shaidurov [2] pointed out that in experiments one also observes disturbances from equilibrium which show a cellular pattern. The object of this paper is to study the stability of equilibrium in terms of spatial disturbances periodic along the cylinder axis. Galerkin's method is used in solving this problem.

1. Equations of the problem. The fluid is considered to fill a horizontal cylindrical cavity within an infinite homogeneous solid mass. A steady temperature gradient A is maintained in the solid for a considerable distance from the cavity, and is directed vertically downwards (the fluid is heated from below). If the magnitude of the gradient is less than the least critical value [3], the fluid will remain in equilibrium. In this case the velocity of the fluid $V_0 = 0$; the temperature gradient in the fluid ∇T_0 and the pressure gradient ∇p_0, in equilibrium, are given by

$$\nabla T_0 = -\frac{2A}{1+\alpha}\gamma = -A'\gamma, \qquad \nabla p_0 = \rho g \beta T_0 \gamma, \qquad \alpha = \frac{\varkappa}{\varkappa_e}$$

In these expressions A' is the equilibrium temperature gradient in the fluid; γ is the unit vector directed vertically upwards; κ and κ_e are the conductivity coefficients of fluid and solid. The small disturbances which arise vary in time according to $e^{-\sigma t}$ where σ is real [3]. At the boundary of stability $\sigma = 0$. The equations for the characteristic motions take the following form:

$$\nabla p = \Delta \mathbf{v} + RT\gamma, \qquad \operatorname{div} \mathbf{v} = 0 \tag{1.1}$$

$$\Delta T = -\mathbf{v}\gamma, \qquad \Delta T_e = 0 \tag{1.2}$$

Here $\mathbf{v}$, T, p, T_e are dimensionless disturbances in velocity, temperature, pressure and solid temperature.

Units of distance, velocity, pressure and temperature are, respectively: a (radius of cylinder), χ / a, $\rho \nu \chi / a^2$, $A' a$. The Rayleigh number $R = g\beta A' a^4 / \nu \chi$ is determined from the equilibrium temperature gradient in the fluid.

The boundary conditions for the dimensionless disturbances are as follows:

$$\begin{aligned} &\mathbf{v},\ T \text{ — are finite} && \text{when } r = 0 \\ &\mathbf{v} = 0, \quad T = T_e, \quad \alpha \frac{\partial T}{\partial r} = \frac{\partial T_e}{\partial r} && \text{when } r = 1 \\ &T_e \to 0 && \text{when } r \to \infty \end{aligned} \tag{1.3}$$

The problem is to look for the characteristic values of the parameter R which determine the critical equilibrium, and the corresponding critical motions of the fluid.

2. Approximation for velocity. Bearing in mind that we solve the problem by Galerkin's method, we approximate the velocity thus:

$$\mathbf{v} = c_1\varphi_1 + \ldots + c_N\varphi_N \tag{2.1}$$

All the functions ϕ_i satisfy the equation of continuity and the boundary conditions of the problem.

Let us introduce Cartesian coordinates, z being along the axis of the cylinder, the axes x and y being in the plane of a section (the x-axis is vertically upwards). Considering periodic disturbances along the z-axis we put

$$v_x = f_1(x, y)\, k \cos kz, \quad v_y = f_2(x, y)\, k \cos kz, \quad v_z = f_3(x, y) \sin kz \tag{2.2}$$

Here k is the wave number of the disturbance. We will look for the functions f_i in polynomial form, subject to vanishing at the surface of the cylinder [4].

$$f_1 = (1 - r^2) \sum_{m,n} a_{mn} x^m y^n, \qquad f_2 = (1 - r^2) \sum_{m,n} b_{mn} x^m y^n$$

$$f_3 = (1 - r^2) \sum_{m,n} c_{mn}\, x^m y^n \tag{2.3}$$

It follows from the equation of continuity

$$\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + f_3 = 0$$

therefore coefficients a_{mn}, b_{mn} and c_{mn} satisfy the expressions

$$(m + 1)[a_{m-1,n} - a_{m+1,n} + a_{m+1,n-2}] + (n + 1)\,[b_{m,n-1} - b_{m,n+1} + \\ + b_{m-2,n+1}] + c_{m-2,n} + c_{m,n-2} - c_{mn} = 0 \qquad (2.4)$$

Let us confine ourselves to $m + n \leqslant 2$. Formulas (2.3) will then contain eighteen unknown coefficients. In view of (2.4) these coefficients will be connected through thirteen relations. There are therefore five unknown coefficients, which allows a system of five basic vectorial functions ϕ_i to be constructed. The choice of such functions is evidently not single-valued. If, however, we consider symmetry, the following system of functions appears to be convenient:

$$\varphi_1 = \begin{cases} (1 - r^2)\, yk \cos kz, \\ -(1 - r^2)\, xk \cos kz, \\ 0 \end{cases} \qquad \varphi_2 = \begin{cases} (1 - r^2)(1 - x^2 - 5y^2)\, k \cos kz \\ 4(1 - r^2)\, xyk \cos kz \\ 0 \end{cases}$$

$$\varphi_3 = \begin{cases} (1 - r^2)^2\, k \cos kz, \\ 0 \\ 4(1 - r^2)\, x \sin kz, \end{cases} \qquad \varphi_4 = \begin{cases} 4(1 - r^2)\, xyk \cos kz \\ (1 - r^2)(1 - 5x^2 - y^2)\, k \cos kz \\ 0 \end{cases}$$

$$\varphi_5 = \begin{cases} 0 \\ (1 - r^2)^2\, k \cos kz \\ 4(1 - r^2)\, y \sin kz \end{cases}$$

The basic motions are illustrated in Fig. 1. The critical motions of the fluid will thus be the superposition of these five basic motions

$$\mathbf{v} = c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5 \qquad (2.5)$$

3. Solution of the problem. Let us find the temperature in the fluid and in the solid. To do this we must substitute the velocity (2.5) into the conductivity equation (1.2) and solve with boundary conditions (1.3). The temperature in the fluid is

$$T = -k \cos kz \{[A_0 + A_2r^2 + A_4r^4 + aI_0(kr)] + c_1[B_1r + B_3r^3 + bI_1(kr)] + \\ + 2\,[D_2r^2 + D_4r^4 + dI_2(kr)]\,(c_3 \cos 2\varphi + c_4 \sin 2\varphi)\} \qquad (3.1)$$

in which

$$A_0 = -\frac{1}{k^6}\,[(k^4 - 8k^2 + 64)\, c_2 + \\ + (k^4 - 16k^2 + 192)\, c_3]$$

$$A_2 = \frac{2}{k^4}\left[(k^2-8)\,c_2 + (2k^2-24)\,c_3\right]$$

$$A_4 = -\frac{1}{k^2}(c_2 + 3c_3)$$

$$B_1 = -\frac{1}{k^4}(k^2-8), \qquad B_3 = D_4 = \frac{1}{k^2}$$

$$D_2 = -\frac{1}{k^4}(k^2-12) \tag{3.2}$$

$$a = \frac{4}{k^6 w_0}\{2\left[(k^2+8)\,K_0' - 4\alpha k K_0\right] c_2 + \left[(8k^2+48)\,K_0' - \alpha\,(k^3+24k)\,K_0\right] c_3\}$$

$$b = \frac{2}{k^5 w_1}\left[-4k\,K_1' + \alpha\,(k^2+4)\,K_1\right]$$

$$d = \frac{2}{k^5 w_1}\left[-6kK_2' + \alpha\,(k^2+12)\,K_2\right]$$

$$w_i = I_i K_i' - \alpha I_i' K_i$$

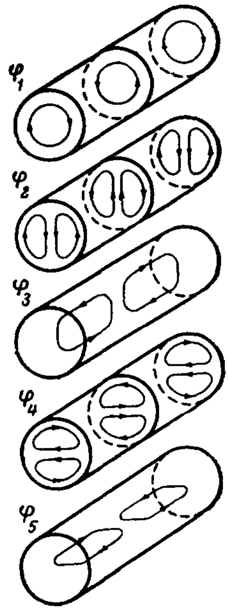

Fig. 1.

In these formulas I_i and K_i are Bessel functions of imaginary argument; in the definition of the coefficients (3.2) the wave number k is the argument. In Formula (3.1) the coefficients c_i are as yet undetermined. The temperature T_e in the solid is no longer required for the calculations, and it is therefore not introduced here.

In order to determine the coefficients c_i we multiply the first of Equations (1.1) by ϕ_i and integrate over the volume of one "nucleus" or "cell" (i.e. over a section of the cylinder along z between the limits 0 to $\lambda = 2\pi/k$). We then get a system of five equations

$$\int \Delta \mathbf{v}\varphi_i \, dV + R\int T\gamma\varphi_i \, dV = 0 \tag{3.3}$$

The integral containing ∇p is equal to zero, a point which is evident if we integrate by parts.

If we substitute the velocity (2.5) and the temperature (3.1) into (3.3) we arrive at a system of five homogeneous linear equations for determining the coefficients c_i. The determinant of this system has the following elements which are nonzero: a_{11}, a_{22}, a_{33}, $a_{23} = a_{32}$, a_{44}, a_{55}, $a_{45} = a_{54}$. The elements of the determinant are very cumbersome functions of the Rayleigh number R, the wave number of the disturbance k and the conductivity ratio a. If we equate the determinant to zero we find five critical Rayleigh numbers and five sets of coefficients c_i

which determine the critical motions.

4. Critical gradients and critical motions. One of the roots is determined from the equation $a_{11} = 0$. From it we find the critical Rayleigh number (as will be evident from what follows, it is convenient to call it the second critical R)

$$R_2 = \frac{2k^5(16+k^2)w_1}{K_1'[12k^2I_4+(k^3+8k)I_5]-\alpha K_1[(k^3+44k)I_4+(7k^2+8)I_5]} \tag{4.1}$$

The critical motion corresponding to it is

$$\mathbf{v}_2 = c_1\varphi_1 \tag{4.2}$$

This function represents a motion with circular trajectories lying in planes perpendicular to the cylinder axis. A displacement of $\lambda/2 = \pi/k$ along the axis results in reversal of motion (Fig. 1). When $k = 0$ (plane disturbances $\lambda = \infty$ [1]*)

$$R_2 = \frac{960(1+\alpha)}{2+7\alpha}$$

When k increases (the wavelength decreases) the Rayleigh number increases monotonically; $R_2 \approx 2k^4$ for $k \gg 1$.

The following two critical Rayleigh numbers are found as roots of the quadratic

$$a_{22}a_{33} - a_{23}a_{32} = 0 \tag{4.3}$$

Here

$$a_{22} = \frac{4}{15}(k^4+30k^2) + \frac{R}{5k^6}\Big\{(30k^4-k^6) + \frac{160I_4}{w_0}[8(k^2+6)K_1 + \alpha K_0(k^3+24k)] + \frac{80I_4}{w_2}[-6k^2K_2' + \alpha K_2(k^3+12k)]\Big\}$$

$$a_{33} = \frac{1}{15}(3k^4+30k^2+160) + \frac{R}{15k^4w_0}\{K_1[54(k^3+36k)I_5 + (3k^4+196k^2+32)I_6] + \alpha K_0[12(3k^3+28k)I_4 + (3k^4+52k^2-160)I_5]\}$$

$$a_{23} = a_{32} = \frac{4}{3}k^2 + \frac{R}{30k^4w_0}\{K_1[18(3k^3+88k)I_5 + (3k^4+176k^2+192)I_6] + \alpha K_0[12(3k^3+8k)I_4 + (3k^4+32k^2-960)I_5]\}$$

* In [1] the Rayleigh number R is expressed through the temperature gradient within the solid, and not in the fluid ($R = R_f$) as in the present work. The connection between them is

$$R_f = \frac{2}{1+\alpha}R_e$$

The expressions for the roots of Equation (4.3) R_1 and R_3 ($R_1 < R_3$) are very cumbersome and are not given here. When $k = 0$

$$R_1 = \frac{23040\,(1+\alpha)}{31+41\alpha}$$

On increasing the wave number, R_1 first of all gets less, goes through a minimum at some value of k and then rises; $R_1 \approx 0.75\ k^4$ when $k \gg 1$.

Root R_3 tends to infinity for $k \to 0$ by the following law:

$$R_3 = \frac{23040}{k^2}\frac{1-0{,}5\alpha k^2 \ln(k/2)}{73-120\alpha\ln(k/2)}$$

When k is increased the root R_3 goes through a minimum, increases and $R_3 \approx 2.35\ k^4$ when $k \gg 1$.

Critical motions corresponding to Rayleigh numbers R_1 and R_3 result from the superposition of basic motions ϕ_2 and ϕ_3

$$\mathbf{v}_1 = c_2\varphi_2 + c_3\varphi_3, \qquad \mathbf{v}_3 = c_2'\varphi_2 + c_3'\varphi_3 \tag{4.4}$$

Because of the homogeneity of the problem coefficients, c_2 and c_2' can be considered arbitrary; the weighting ratios c_3/c_2 and c_3'/c_2' depend on k. In the region of k of the order of unity the critical motion $\mathbf{v}_1$ contains, in the main, basic function ϕ_3 with a small admixture of ϕ_2. When $k \to 0$, however, the weighting ratio changes sharply and $\mathbf{v}_1 \to c_2\phi_2$.

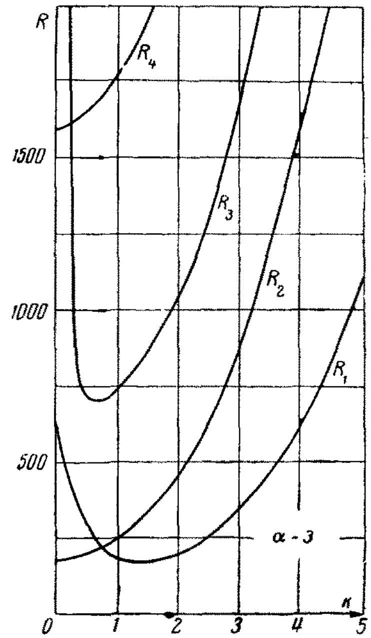

Fig. 2.

Motion $\mathbf{v}_3$ on the other hand consists mainly of ϕ_2 with an admixture of ϕ_3, but when $k \to 0$ it transforms into pure motion ϕ_3 (when $k = 0$ horizontal trajectories correspond to motion ϕ_3 and thus $R_3 \to \infty$). When k is great the basic functions ϕ_2 and ϕ_3 enter the critical motions $\mathbf{v}_1$ and $\mathbf{v}_3$ with approximately equal weight, whilst $c_3/c_2 > 0$, and $c_3'/c_2' < 0$.

The remaining two critical values R_4 and R_5 are found from equation $a_{44}a_{55} - a_{45}a_{54} = 0$; calculation gives

$$R_4 = \frac{12k^5\,(k^6+40k^4+320k^2+1600)}{3k^4+30k^2+160} \times$$

$$\times \frac{w_2}{K_2' \,[16k^2 I_5 + (k^3 + 18k)\, I_6] - \alpha K_2 \,[(k^3 + 82k)\, I_5 + (10k^2 + 36)\, I_6]} \tag{4.5}$$

When $k = 0$

$$R_4 = \frac{23040\,(1+\alpha)}{7 + 17\alpha}$$

When k is increased root R_4 increases monotonically and $R_4 \approx 4\,k^4$ when $k \gg 1$.

The critical motion $\mathbf{v}_4$ is indeed a combination of ϕ_4 and ϕ_5 with definite weighting ratios:

$$\mathbf{v}_4 = c_4\varphi_4 + c_5\varphi_5, \quad \frac{c_5}{c_4} = -\frac{20k^2}{3k^4 + 30k^2 + 160}$$

From this it is evident that when $k \to 0$ $\mathbf{v}_4 \to c_4\phi_4$.

The critical motion $\mathbf{v}_5$ contains only the basic function ϕ_5, so that $\mathbf{v}_5 = c_5'\phi_5$; the trajectories of this motion are horizontal and the corresponding critical Rayleigh number is infinite.

In our approximation, therefore, it has been possible to find five critical motions and the critical temperature gradients corresponding to them. As an example, in Fig. 2, critical Rayleigh numbers are shown as functions of wave number of disturbance for $a = 3$. The character of the spectrum does not alter with variation in a. Shifts in the stability curves with changes in a can be assessed from the limiting values of Rayleigh numbers when $k = 0$ and when $k \gg 1$, derived above. It is evident from Fig. 2 that the motions which are most dangerous from the point of view of upsetting stability are $\mathbf{v}_1$ and $\mathbf{v}_2$; the two lower level spectra correspond to these. On the stability curve $R_2(k)$ the minimum is attained when $k = 0$, i.e. when $\lambda = \infty$ (plane disturbances). On the curve $R_1(k)$ the minimum is attained at some finite value of k, i.e. to the minimum R_{1*} there corresponds a subdivision into nucleus cells of given length; i.e. the picture is similar to what happens in the Rayleigh case of instability in a plane horizontal layer (Benard cells). We give below minimum values of Rayleigh numbers R_{1*} and R_{2*} for several values of a.

α	0	1	10	100	∞
R_{1*}	260	210	134	102	96
R_{2*}	480	213	147	138	137

It is evident that $R_{1*} < R_{2*}$ over the whole range of a, and the values do not differ greatly from each other. The simultaneous appearance, therefore, of both critical motions is quite likely to occur in an experiment. It would appear that a superposition of these critical motions was indeed observed in Shaidurov's tests.

BIBLIOGRAPHY

1. Zhukhovitskii, E.M., Primenenie metoda Galerkina k zadache ob ustoichivosti neravnomerno nagretoi zhidkosti (Application of Galerkin's method to the problem of stability of a nonuniformly heated fluid). *PMM* Vol. 18, No. 2, 1954.

2. Shaidurov, G.F., Teplovaia neustoichivost' zhidkosti v gorizontal'nom tsilindre (Thermal instability of fluid in a horizontal cylinder). *Inzh. Fiz. Zh.* Vol. 4, No. 2, 1961.

3. Sorokin, V.S., Variatsionnyi method v teorii konvektsii (Variational method in convection theory). *PMM* Vol. 17, No. 1, 1953.

4. Zhukhovitskii, E.M., Ob ustoichivosti neravnomerno nagretoi zhidkosti v sharovoi polosti (The stability of a nonuniformly heated fluid in a spherical cavity). *PMM* Vol. 21, No. 5, 1957.

Translated by V.H.B.